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ON THE INTERVAL—ARITHMETIC PRESENTATION OF THE RANGE OF A CLASS OF MONOTONE FUNCTIONS OF MANY VARIABLES

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Abstract: We consider a technique for the exact presentation of the range of a certain class of monotone functions of many variables by means of interval-arithmetic expressions involving nonstandard interval arithmetic operations.

1. Introduction.

By $I(D)$, $D\subset R^n$, we denote the set of all intervals (interval vectors) on D. For the interval $X\in I(D)$ we shall write $X=[\underline{x},\overline{x}]=(\ [\underline{x}_i,\overline{x}_i]\ )_{i=1}^{n}=(\ X_i\ )_{i=1}^{n}$.

Let $f:D\ R$ be a real-valued function of n variables defined in D. Denote by $f^*(X)$ the range of f on an interval $X\in I(D)$, i.e.

$$f^*(X)=\{f(x)\ :\ x\in X\}$$
$$=\{f(x_1,x_2,\dots,x_n)\ :\ x_1\in X_1,\ x_2\in X_2,\dots,\ x_n\in X_n\},$$

and by $f(X)$ the optimal interval hull of $f^*(X)$ [10], i.e.

$$f(X)\ =\ [\inf f^*(X)\ ,\ \sup f^*(X)].$$

If f is a continuous function then obviously $f^*(X)=f(X)$ holds true.

We next define a class of functions [10] satisfying certain monotonicity properties.

Definition 1. We shall say that $x=(x_1,x_2,\dots,x_n)$, $y=(y_1,y_2,\dots,y_n)\in R^n$ are partially ordered and write $x\circ y$, if $x_i \circ_i y_i$ for each $i=1,2,\ldots,n$, where $\circ_i \in \{\le,\ge\}$, $\le$ and $\ge$ being the usual order relations in R.

Denote by $\mathfrak{O}$ the class of all order relations $\circ = (\circ_1,\circ_2,\ldots,\circ_n)$; there are obviously 2^n such relations in $\mathfrak{O}$.

Definition 2. A function $f: D \to R$, $D \subset R^n$, belongs to $M(D)$, if there is a partial ordering $\circ = (\circ_1,\circ_2,\ldots,\circ_n) \in \mathfrak{O}$, such that for each $x, y \in D$, $x \circ y$ implies $f(x) \le f(y)$.

Remark. A function $f \in M(D)$ is called in [10] obligatory (mandatory) monotone. Roughly speaking, a function $f \in M(D)$ can be differently monotone with respect to the different components, that is f may be isotone (monotonically increasing) with respect to some components and antitone (monotonically decreasing) with respect to other components.

The exact presentation of the range of a function over compact intervals is an important precondition for the construction of effective inclusion algorithms. The usual interval arithmetic [2,9] allows us to obtain in general (inner and outer) inclusions for the ranges (see [1,4,5]), that can be sometimes very rough. Exact presentations for the ranges (which are special cases of inclusions) can be seldomly obtained by means of usual interval arithmetic. A special situation when these inclusions degenerate into exact presentations is considered in [9]. This is the situation when the function is equally monotone with respect to all components, that is either isotone with respect to all components or antitone with respect to all x_i's. However, this is only a very special case of a function from $M(D)$.

The extended interval arithmetic [3,6,7,8], which is shortly summarized in Sec. 2 of this paper, is a suitable tool to extend the possibilities for the exact presentation of the range of an arbitrary function belonging to $M(D)$. By choosing the domain D in a suitable way, this tool can be further applied to an important set of functions, which is often used in Numerical Analysis [11]. These functions can be defined by algebraic expressions involving arithmetic operations and standard functions of certain types.

Using extended interval arithmetic operations we can find exact presentations for the range of functions belonging to $M(D)$ in the form of interval-arithmetic expressions. Our results can be considered as generalizations of Propositions 1-4 in [7].

2. Extended Interval Arithmetic.

Denote by R the set of real numbers and by $I(R)$ the set of all compact intervals on R. Denote by $\underline{x}$ and $\overline{x}$, $\underline{x}\le\overline{x}$, the end-points of $X\in I(R)$, i.e. $X=[\underline{x},\overline{x}]$. The width of X will be denoted by $w(X)=\overline{x}-\underline{x}$. For our purposes, we shall make use of one more end-point-wise presentation of an interval X. By x^{-0} and x^{+0} we shall mean the end-points of X such that $|x^{+0}|<|x^{-0}|$; that is x^{+0} is the end-point of X which is closer to zero than x^{-0}. More precisely, we define

$$x^{+0}=\begin{cases}\underline{x}, & |\underline{x}|\le|\overline{x}|,\\ \overline{x} & \text{otherwise};\end{cases}\qquad x^{-0}=\begin{cases}\overline{x}, & |\underline{x}|\le|\overline{x}|,\\ \underline{x} & \text{otherwise}.\end{cases}$$

We then write $X=[x^{-0}\vee x^{+0}]$. In this notation it is not explicitly pointed out which end-point is the left one and which is the right one.

The extended interval arithmetic operations are introduced as follows [3,6,7,8]. For $X, Y \in I(R)$, $X=[\underline{x},\overline{x}]=[x^{+0} \vee x^{-0}]$, $Y=[\underline{y},\overline{y}]=[y^{+0} \vee y^{-0}]$ we define:

$$X + Y = [\underline{x} + \underline{y}\ ,\ \overline{x} + \overline{y}];$$

$$X - Y = [\underline{x} - \overline{y}\ ,\ \overline{x} - \underline{y}];$$

$$X \times Y = [x^{-0}y^{-0} \vee x^{+0}y^{+0}] \text{ if } 0 \notin X,Y;$$

$$X \,/\, Y = [x^{-0}/y^{+0} \vee x^{+0}/y^{-0}] \text{ if } 0 \notin X,Y;$$

$$X \mathbin{\boldsymbol{+}} Y = [\underline{x} + \overline{y} \vee \overline{x} + \underline{y}];$$

$$X \mathbin{\boldsymbol{-}} Y = [\underline{x} - \underline{y} \vee \overline{x} - \overline{y}];$$

$$X \mathbin{\mathbf{x}} Y = [x^{-0}y^{+0} \vee x^{+0}y^{-0}] \text{ if } 0 \notin X,Y;$$

$$X \mathbin{\boldsymbol{/}} Y = [x^{-0}/y^{-0} \vee x^{+0}/y^{+0}] \text{ if } 0 \notin X,Y.$$

Multiplication and division for $0 \in X,Y$ will not be used in the sequence and therefore will not be introduced here. For this special case consult for example [6].

Furthermore, we make use of the following relations [8]:

Lemma 1. Let $\alpha,\beta,\gamma,\delta \in R$. Then

a) $$[(\alpha+\beta)\vee(\gamma+\delta)] = \begin{cases} [\alpha\vee\gamma] + [\beta\vee\delta] & \text{if } (\alpha-\gamma)(\beta-\delta)\geq 0, \\ [\alpha\vee\gamma] \mathbin{\boldsymbol{+}} [\beta\vee\delta] & \text{otherwise;} \end{cases}$$

b) $$[(\alpha-\beta)\vee(\gamma-\delta)] = \begin{cases} [\alpha\vee\gamma] \mathbin{\boldsymbol{-}} [\beta\vee\delta] & \text{if } (\alpha-\gamma)(\beta-\delta)\geq 0, \\ [\alpha\vee\gamma] - [\beta\vee\delta] & \text{otherwise;} \end{cases}$$

Lemma 2. Let $\alpha,\beta,\gamma,\delta \in R$ such that $\alpha\gamma>0$ and $\beta\delta>0$. Then

a) $$[(\alpha\beta)\vee(\gamma\delta)] = \begin{cases} [\alpha\vee\gamma] \times [\beta\vee\delta] & \text{if } (|\alpha|-|\gamma|)(|\beta|-|\delta|)\geq 0, \\ [\alpha\vee\gamma] \mathbin{\mathbf{x}} [\beta\vee\delta] & \text{otherwise;} \end{cases}$$

b) $$[(\alpha/\beta)\vee(\gamma/\delta)]=\begin{cases}[\alpha\vee\gamma]\ /\ [\beta\vee\delta] \text{ if } (|\alpha|-|\gamma|)(|\beta|-|\delta|)\geq 0,\\ [\alpha\vee\gamma]\ /\ [\beta\vee\delta] \text{ otherwise;}\end{cases}$$

We shall also make use of the following results for functions of one variable. In the following four propositions [7] it is assumed that f is a monotone function on D, $D\subset R$, and f is representable as $f=f_1 * f_2$, $*=+,-,\times,/$, resp., according to the proposition, and f_1, f_2 are also monotone on D. The following notations are used concerning the intervals $f_i(X)$, $i=1,2$, $X=[\underline{x},\overline{x}]$:

$$d(f_i(X))=f_i(\underline{x})-f_i(\overline{x}),$$

$$r(f_i(X))=|f_i(\underline{x})|-|f_i(\overline{x})|,\ i=1,2.$$

Proposition 1. If $f=f_1+f_2$ then for every $X\in I(D)$

$$f(X) = \begin{cases} f_1(X) + f_2(X) & \text{if } d(f_1(X)).d(f_2(X)) \geq 0,\\ f_1(X) \boldsymbol{+} f_2(X) & \text{otherwise;}\end{cases}$$

Proposition 2. If $f=f_1-f_2$ then for every $X\in I(D)$

$$f(X) = \begin{cases} f_1(X) \boldsymbol{-} f_2(X) & \text{if } d(f_1(X)).d(f_2(X)) \geq 0,\\ f_1(X) - f_2(X) & \text{otherwise.}\end{cases}$$

Proposition 3. If $f=f_1\times f_2$ and $f_i(x)\neq 0$ for every $x\in X$, $i=1,2$, then for every $X\in I(D)$

$$f(X) = \begin{cases} f_1(X) \times f_2(X) & \text{if } r(f_1(X)).r(f_2(X)) \geq 0,\\ f_1(X)\ \mathbf{x}\ f_2(X) & \text{otherwise.}\end{cases}$$

Proposition 4. If $f=f_1/f_2$ and $f_i(x)\neq 0$ for every $x\in X$, $i=1,2$ then for every $X\in I(D)$

$$f(X) = \begin{cases} f_1(X) \ /\ f_2(X) & \text{if } d(f_1(X)).d(f_2(X)) \geq 0,\\ f_1(X) \ /\ f_2(X) & \text{otherwise.}\end{cases}$$

Remark 1. The condition $d(f_1(X)).d(f_2(X))\geq 0$ (resp.

$r(f_1(X)).r(f_2(X))\ge 0$) means that the functions f_1 and f_2 (resp. $|f_1|$ and $|f_2|$) are both increasing or both decreasing on X.

Remark 2. A generalization of Proposition 3 in the situation when $f = f_1\times f_2\times\ldots\times f_n$ and all functions are either increasing or decreasing on X may be obtained in natural way, namely $f(X)=f_1(X)\times f_2(X)\times\ldots\times f_n(X)$.

3. Exact interval-arithmetic presentation of ranges of functions of many variables belonging to $M(D)$.

We shall now propose a technique to obtain interval-arithmetic expressions for the interval $f(X)$ in the situation when $f\in M(D)$, $D\subset R^n$, and f is representable as a sum, difference, product or quotient of two other functions, i.e. $f=f_1 * f_2$, $*\in\{+,-,\times,/\}$.

Let $f\in M(D)$. We define two sets of indices $I,J\subseteq \{1,2,\ldots,n\}\cup\emptyset$ with $I\cap J=\emptyset$ and $I\cup J=\{1,2,\ldots,n\}$. According to Definition 2, there exists a partial ordering $\circ=(\circ_1,\circ_2,\ldots,\circ_n)$, corresponding to f such that $\circ_i\in\{\le,\ge\}$, $i=1,2,\ldots,n$. We define I as the set of all indices i, such that $\circ_i=\le$, and J as the set of all indices j, such that $\circ_j=\ge$.

Let $X=([\underline{x}_i,\overline{x}_i])_{i=1}^{n}\in I(D)$. We define the real vectors $u(f;X)=(u_1,u_2,\ldots,u_n)$ and $v(f;X)=(v_1,v_2,\ldots,v_n)$ in the following manner:

$$u_i = \begin{cases} \underline{x}_i & \text{for } i\in I, \\ \overline{x}_i & \text{for } i\in J; \end{cases} \qquad v_i = \begin{cases} \underline{x}_i & \text{for } i\in J, \\ \overline{x}_i & \text{for } i\in I; \end{cases}$$

$i=1,2,\ldots,n$.

Then we can present the interval f(X) in the form

(1) $$f(X) = [f(u(f;X)) \ , \ f(v(f;X))] \ .$$

Denote by C the convex hull of the vectors u(f;X) and v(f;X), i.e.

$$C = \mathrm{co}\{u(f;X),\ v(f;X)\}$$
$$= \{tu(f;X) + (1-t)v(f;X) \ : \ t\in[0,1]\}.$$

Obviously, $C \subset X$ holds true.

Let us assume that $f=f_1 * f_2$, $*\in\{+,-,\times,/\}$. For the functions f_1 and f_2 we introduce the intervals

(2) $$f_i(C)=[f_i(u(f;X)) \vee f_i(v(f;X))],\ i=1,2,$$

and also the notations concerning these intervals

(3) $$d(f_i(C)) = f_i(u(f;X)) - f_i(v(f;X));$$

(4) $$r(f_i(C)) = |f_i(u(f;X))| - |f_i(v(f;X))|, \quad i=1,2.$$

Theorem. Let $f\in M(D)$, $f=f_1 * f_2$, $*\in\{+,-,\times,/\}$ and the convex hull C, the intervals $f_i(C)$ and $d(f_i(C))$, $r(f_i(C))$, $i=1,2$, are defined as above. Then for every $X\in I(D)$ the following exact interval-arithmetic presentations for the optimal interval hull f(X) hold true:

a) $f= f_1+f_2$:

$$f(X)=\begin{cases} f_1(C) + f_2(C) \text{ if } d(f_1(C)).d(f_2(C)) \geq 0, \\ f_1(C) \boldsymbol{+} f_2(C) \text{ otherwise;} \end{cases}$$

b) $f=f_1-f_2$:

$$f(X)=\begin{cases} f_1(C) \boldsymbol{-} f_2(C) \text{ if } d(f_1(C)).d(f_2(C)) \geq 0, \\ f_1(C) - f_2(C) \text{ otherwise;} \end{cases}$$

c) $f=f_1\times f_2$, $f_i(x)\neq 0$ for every $x\in X$, $i=1,2$:

$$f(X)=\begin{cases} f_1(C) \times f_2(C) & \text{if } r(f_1(C)).r(f_2(C)) \geq 0, \\ f_1(C) \ \mathbf{x}\ f_2(C) & \text{otherwise;} \end{cases}$$

d) $f=f_1/f_2$, $f_i(x)\neq 0$ for every $x\in X$, $i=1,2$:

$$f(X)=\begin{cases} f_1(C) \ / \ f_2(C) & \text{if } r(f_1(C)).r(f_2(C)) \geq 0, \\ f_1(C) \ / \ f_2(C) & \text{otherwise.} \end{cases}$$

Proof.

a) Using (1) we get for $f=f_1+f_2$

$$\begin{aligned} f(X)&=[f(u(f;X)),f(v(f;X))] \\ &=[(f_1+f_2)(u(f;X)) \ , \ (f_1+f_2)(v(f;X))] \\ &=[f_1(u(f;X))+f_2(u(f;X)) \ , \ f_1(v(f;X))+f_2(v(f;X))]. \end{aligned}$$

From Lemma 1 a) it follows that

$$f(X)=[f_1(u(f;X))\vee f_1(v(f;X))] + [f_2(u(f;X))\vee f_2(v(f;X))]$$
$$\text{if } (f_1(u(f;X))-f_1(v(f;X))).(f_2(u(f;X))-f_2(v(f;X)))\geq 0$$

and

$$f(X)=[f_1(u(f;X))\vee f_1(v(f;X))] \ \boldsymbol{+} \ [f_2(u(f;X))\vee f_2(v(f;X))]$$
$$\text{if } (f_1(u(f;X))-f_1(v(f;X))).(f_2(u(f;X))-f_2(v(f;X)))<0.$$

Using (2) and (3) we obtain finally

$$f(X)=\begin{cases} f_1(C) + f_2(C) & \text{if } d(f_1(C)).d(f_2(C)) \geq 0, \\ f_1(C) \ \boldsymbol{+}\ f_2(C) & \text{otherwise.} \end{cases}$$

The verification of b) we get similarly using Lemma 1 b), (1), (2) and (3).

c) From (1) we obtain

$$\begin{aligned} f(X)&=[f(u(f;X)) \ , \ f(v(f;X))] \\ &=[(f_1\times f_2)(u(f;X)) \ , \ (f_1\times f_2)(v(f;X))] \\ &=[f_1(u(f;X))\times f_2(u(f;X)) \ , \ f_1(v(f;X))\times f_2(v(f;X))]. \end{aligned}$$

Using Lemma 2 a) it follows that

$$f(X)=[f_1(u(f;X))\vee f_1(v(f;X))] \times [f_2(u(f;X))\vee f_2(v(f;X))] \text{ if}$$

$$(|f_1(u(f;X))|-|f_1(v(f;X))|).(|f_2(u(f;X))|-|f_2(v(f;X))|)\geq 0$$

and

$$f(X)=[f_1(u(f;X))\vee f_1(v(f;X))] \boldsymbol{\times} [f_2(u(f;X))\vee f_2(v(f;X))]$$

otherwise.

Using (2) and (4) we obtain

$$f(X)=\begin{cases} f_1(C) \times f_2(C) \text{ if } r(f_1(C)).r(f_2(C)) \geq 0, \\ f_1(C) \boldsymbol{\times} f_2(C) \text{ otherwise;} \end{cases}$$

d) can be proved similarly using Lemma 2 b), (1), (2) and (4).

Remark 3. The intervals $f_i(C)=[f_i(u(f;X)) \vee f_i(v(f;X))]$ $i=1,2$, can be presented in the form

$$f_i(C)=[f_i(tu(f;X)+(1-t)v(f;X))|_{t=1} \quad \vee$$

$$f_i(tu(f;X)+(1-t)v(f;X))|_{t=0}],$$

i.e. the end-points of the intervals $f_i(C)$ are equal to the values of the functions $f_i(tu(f;X)+(1-t)v(f;X))$ at the end-points of the interval $[0,1]$. We can consider f_i over the convex hull C as functions of one variable t on the interval $[0,1]$. We denote

$$f_i(tu(f;X)+(1-t)v(f;X))=\hat{f}_i(t), \ i=1,2.$$

If $\hat{f}_i$, $i=1,2$, are monotone (and continuous) with respect to t then we obtain

$$f_i(C)=\{f_i(tu(f;X)+(1-t)v(f;X) : t\in[0,1]\}$$
$$=\hat{f}_i([0,1]), \ i=1,2.$$

Hence, the intervals $f_i(C)$ are equal to the ranges of the functions $\hat{f}_i(t)$ on $[0,1]$. In the situation when it is possible to present $\hat{f}_i(t)$ as $\hat{f}_i(t)=g_{i1}(t)*g_{i2}(t)$ for $*\in\{+,-,\times,/\}$ we can then obtain interval-arithmetic presentations for $f_i(C)$ using Propositions 1-4 from Sec. 2. Thus, the problem is reduced to obtain the ranges of the functions $g_{i1}(t)$, $g_{i2}(t)$ etc.

4. An example.

Let us consider the following function of n variables

$$f(x_1,x_2,\ldots,x_n)=(x_k-x_1)\ldots(x_k-x_{k-1})(x_k-x_{k+1})\ldots(x_k-x_n), \tag{5}$$

where $1<k<n$. Let the real intervals X_1, X_2, ..., X_n be such that $X_i\cap X_j=\emptyset$ for $i\neq j$, $i,j=1,2,\ldots,n$, $X_1<X_2<\ldots<X_n$ and $x_i\in X_i$ for $i=1,2,\ldots,n$. Denoting by $x=(x_1,x_2,\ldots,x_n)$ and $X=(X_1,X_2,\ldots,X_n)$ we shall find interval-arithmetic expressions for the range of the function (5), i.e. for the range

$$f(X) = \{f(x) : x\in X\}$$

This problem will be solved under the following assumptions:

i) the integer (n-k) is odd, $\sum_{\substack{l=1\\ l\neq k}}^{n}\prod_{\substack{j=1\\ j\neq l\\ j\neq k}}^{n}(x_k-x_j)>0$;

ii) the integer (n-k) is even, $\sum_{\substack{l=1\\ l\neq k}}^{n}\prod_{\substack{j=1\\ j\neq l\\ j\neq k}}^{n}(x_k-x_j)>0$;

iii) the integer (n-k) is odd, $\sum_{\substack{l=1\\ l\neq k}}^{n}\prod_{\substack{j=1\\ j\neq l\\ j\neq k}}^{n}(x_k-x_j)<0$;

iv) the integer (n-k) is even, $\sum_{\substack{l=1\\ l\neq k}}^{n}\prod_{\substack{j=1\\ j\neq l\\ j\neq k}}^{n}(x_k-x_j)<0$.

We shall consider in details the situation in case i).

We denote

$$f_1(x)=(x_k-x_1)(x_k-x_2)\ldots(x_k-x_{k-1}),$$

$$f_2(x)=(x_k-x_{k+1})(x_k-x_{k+2})\ldots(x_k-x_n).$$

Then $f(x)=f_1(x)\times f_2(x)$ holds true. To apply the Theorem c) from Section 3 we have to verify the assumptions of this Theorem. For this purpose, we compute the partial derivatives $\partial f/\partial x_i$, $i=1,2,\dots,n$. Using the assumptions i) it is easy to see that

$$\partial f/\partial x_i \begin{cases} >0 \text{ for } i\le k-1, \\ <0 \text{ for } i\ge k+1; \end{cases} \tag{6}$$

$$\partial f/\partial x_k > 0. \tag{7}$$

From (6) and (7) we get for the real vectors $u(f;X)$ and $v(f;X)$,

$$u(f;X)=(\underline{x}_1,\underline{x}_2,\dots,\underline{x}_{k-1},\underline{x}_k,\overline{x}_{k+1},\dots,\overline{x}_n),$$

$$v(f;X)=(\overline{x}_1,\overline{x}_2,\dots,\overline{x}_{k-1},\overline{x}_k,\underline{x}_{k+1},\dots,\underline{x}_n).$$

Let $C=\mathrm{co}\{u(f;X),v(f;X)\}=\{tu(f;X)+(1-t)v(f;X) : t\in[0,1]\}$. For $f_1(C)$ we obtain

$$\begin{aligned} f_1(C)=\{&(\xi_k-\xi_1)(\xi_k-\xi_2)\dots(\xi_k-\xi_{k-1}) \ , \ \xi_1=t\underline{x}_1+(1-t)\overline{x}_1, \\ &\xi_2=t\underline{x}_2+(1-t)\overline{x}_2, \ \dots \ , \ \xi_{k-1}=t\underline{x}_{k-1}+(1-t)\overline{x}_{k-1}, \\ &\xi_k=t\underline{x}_k+(1-t)\overline{x}_k \ : \ t\in[0,1] \ \} \\ =\{&(t(\underline{x}_k-\underline{x}_1)+(1-t)(\overline{x}_k-\overline{x}_1))(t(\underline{x}_k-\underline{x}_2)+(1-t)(\overline{x}_k-\overline{x}_2))\times\dots \\ &\times(t(\underline{x}_k-\underline{x}_{k-1})+(1-t)(\overline{x}_k-\overline{x}_{k-1})) \ : \ t\in[0,1]\} \\ =\{&(t(w(X_1)-w(X_k))+(\overline{x}_k-\overline{x}_1))(t(w(X_2)-w(X_k))+(\overline{x}_k-\overline{x}_2))\times \\ &\dots\times(t(w(X_k)-w(X_{k-1}))+(\overline{x}_k-\overline{x}_{k-1})) \ : \ t\in[0,1]\} \\ =\{&\hat{f}_1(t) \ : \ t\in[0,1]\} = \hat{f}_1([0,1]). \end{aligned}$$

Let $\{i_1,i_2,\dots,i_p\}$ be a subset of the set of indices $\{1,2,\dots,k-1\}$, such that $w(X_{i_l})\ge w(X_k)$, $l=1,2,\dots,p$, and $\{j_1,j_2,\dots,j_q\}$ be the subset, for which $w(X_{j_l})<w(X_k)$, $l=1,2,\dots,q$, holds true. Let us further present $\hat{f}_1(t)=g_{11}(t)\times g_{12}(t)$, where

$$g_{11}(t)=\prod_{l=1}^{p}(t(w(X_{i_l})-w(X_k))+(\bar{x}_k-\bar{x}_{i_l})),$$

$$g_{12}(t)=\prod_{l=1}^{q}(t(w(X_{j_l})-w(X_k))+(\bar{x}_k-\bar{x}_{j_l})).$$

For $t\in[0,1]$ the function $g_{11}(t)$ is increasing and $g_{12}(t)$ is decreasing. Using Proposition 3 we obtain

$$f_1(C)=\hat{f}_1([0,1])=g_{11}([0,1])\ \mathbf{x}\ g_{12}([0,1]).$$

The function $g_{11}(t)$ is a product of p increasing functions over the interval [0,1] which do not vanish in [0,1] (see Remark 2), from where it follows

$$g_{11}([0,1])=(X_k-X_{i_1})\times(X_k-X_{i_2})\times\ldots\times(X_k-X_{i_p}).$$

Since the function $g_{12}(t)$ is a product of q decreasing and not vanishing over [0,1] functions, we obtain

$$g_{12}([0,1])=(X_k-X_{j_1})\times(X_k-X_{j_2})\times\ldots\times(X_k-X_{j_q}).$$

Therefore,

$$f_1(C)=((X_k-X_{i_1})\times\ldots\times(X_k-X_{i_p}))\ \mathbf{x}\ ((X_k-X_{j_1})\times\ldots\times(X_k-X_{j_q})).$$

Let us now get an interval-arithmetic presentation for $f_2(C)$. Analogously, we have

$$\begin{aligned}f_2(C)=\{&(\xi_k-\xi_{k+1})\ldots(\xi_k-\xi_n)\ ,\ \xi_k=t\underline{x}_k+(1-t)\bar{x}_k,\\ &\xi_{k+1}=t\bar{x}_{k+1}+(1-t)\underline{x}_{k+1},\ldots,\xi_n=t\bar{x}_n+(1-t)\underline{x}_n:\qquad t\in[0,1]\}\\ =\{&(t(\underline{x}_k-\bar{x}_{k+1})+(1-t)(\bar{x}_k-\underline{x}_{k+1}))\times\ldots\times\\ &\qquad(t(\underline{x}_k-\bar{x}_n)+(1-t)(\bar{x}_k-\underline{x}_n))\ :\ t\in[0,1]\}\\ =\{&(t(-w(X_{k+1})-w(X_k))+(\bar{x}_k-\underline{x}_{k+1}))\times\ldots\times\\ &\qquad(t(-w(X_n)-w(X_k))+(\bar{x}_k-\underline{x}_n))\ :\ t\in[0,1]\}\\ =\{&\hat{f}_2(t)\ :\ t\in[0,1]\}=\hat{f}_2([0,1]).\end{aligned}$$

The function $\hat{f}_2(t)$ represents a product of n-k decreasing on [0,1] functions, which do not vanish in [0,1], i.e.

$$\hat{f}_2(t)=\prod_{i=1}^{n-k-1} g_{2i}(t),$$

$$g_{2i}(t)=t(-w(X_{k+i})-w(X_k))+(\overline{x}_k-\underline{x}_{k+i}),\quad i=1,\dots,n-k.$$

From Proposition 3 (and Remark 2) we obtain

$$f_2(C)=g_{21}([0,1])\times g_{22}([0,1])\times\dots\times g_{2n-k}([0,1])$$

$$=(X_k-X_{k+1})\times(X_k-X_{k+2})\times\dots\times(X_k-X_n),$$

since $g_{2i}([0,1])=X_k-X_{k+i}$, $i=1,2,\dots,n-k$ holds true.

Finally, we need the values $r(f_1(C))$ and $r(f_2(C))$ as they are defined by (4):

$$r(f_1(C))=|f_1(u(f{:}X))|-|f_1(v(f;X))|$$

$$=|(\underline{x}_k-\underline{x}_1)\dots(\underline{x}_k-\underline{x}_{k-1})| - |(\overline{x}_k-\overline{x}_1)\dots(\overline{x}_k-\overline{x}_{k-1})|$$

$$=(\underline{x}_k-\underline{x}_1)\dots(\underline{x}_k-\underline{x}_{k-1}) - (\overline{x}_k-\overline{x}_1)\dots(\overline{x}_k-\overline{x}_{k-1});$$

$$r(f_2(C))=|f_2(u(f{:}X))|-|f_2(v(f;X))|$$

$$=|(\underline{x}_k-\overline{x}_{k+1})\dots(\underline{x}_k-\overline{x}_n)| - |(\overline{x}_k-\underline{x}_{k+1})\dots(\overline{x}_k-\underline{x}_n)|$$

$$=(\underline{x}_k-\overline{x}_{k+1})\dots(\underline{x}_k-\overline{x}_n) - (\overline{x}_k-\underline{x}_{k+1})\dots(\overline{x}_k-\underline{x}_n).$$

It is not difficult to verify that $r(f_2(C))>0$ holds true. Using Theorem c) we obtain

$$f(X)=\begin{cases} f_1(C)\times f_2(C) & \text{if } r(f_1(C))>0,\\ f_1(C)\ \mathbf{x}\ f_2(C) & \text{otherwise.}\end{cases}$$

Replacing the intervals $f_1(C)$ and $f_2(C)$ with the corresponding interval-arithmetic expressions we obtain the following final result:

$$f(X)=(((X_k\,\mathbf{-}\,X_{i_1})\times\dots\times(X_k\,\mathbf{-}\,X_{i_p}))\ \mathbf{x}\ ((X_k\,\mathbf{-}\,X_{j_1})\times\dots\times(X_k\,\mathbf{-}\,X_{j_q})))\times$$
$$((X_k-X_{k+1})\times\dots\times(X_k-X_n)) \text{ if } r(f_1(C))>0,$$

$$f(X)=(((X_k\,\mathbf{-}\,X_{i_1})\times\dots\times(X_k\,\mathbf{-}\,X_{i_p}))\ \mathbf{x}\ ((X_k\,\mathbf{-}\,X_{j_1})\times\dots\times(X_k\,\mathbf{-}\,X_{j_q})))\ \mathbf{x}$$
$$((X_k-X_{k+1})\times\dots\times(X_k-X_n)) \text{ if } r(f_1(C))<0.$$

The remaining three cases can be treated analogously .

ii)
$$f(X)=((X_k-X_1)\times(X_k-X_2)\times\ldots\times(X_k-X_{k-1}))\times$$
$$(((X_k\boldsymbol{-}X_{k+i_1})\times\ldots\times(X_k\boldsymbol{-}X_{k+i_p}))\ \boldsymbol{\times}\ ((X_k\boldsymbol{-}X_{k+j_1})\times\ldots\times(X_k\boldsymbol{-}X_{k+j_q})))$$
$$\text{if } r_2<0,$$

$$f(X)=((X_k-X_1)\times(X_k-X_2)\times\ldots\times(X_k-X_{k-1}))\ \boldsymbol{\times}$$
$$(((X_k\boldsymbol{-}X_{k+i_1})\times\ldots\times(X_k\boldsymbol{-}X_{k+i_p}))\ \boldsymbol{\times}\ ((X_k\boldsymbol{-}X_{k+j_1})\times\ldots\times(X_k\boldsymbol{-}X_{k+j_q})))$$
$$\text{if } r_2>0,$$

where $\{i_1,i_2,\ldots,i_p\}$ is a subset of $\{1,2,\ldots,n-k\}$ such that $w(X_{k+i_l})\geq w(X_k)$, $l=1,2,\ldots,p$, and $\{j_1,j_2,\ldots,j_q\}$ is this subset of $\{1,2,\ldots,n-k\}$, for which $w(X_{k+j_l})<w(X_k)$, $l=1,2,\ldots,q$ holds,

and
$$r_2=|(\underline{x}_k-\underline{x}_{k+1})(\underline{x}_k-\underline{x}_{k+2})\ldots(\underline{x}_k-\underline{x}_n)|-$$
$$|(\overline{x}_k-\overline{x}_{k+1})(\overline{x}_k-\overline{x}_{k+2})\ldots(\overline{x}_k-\overline{x}_n)|.$$

iii)
$$f(X)=((X_k-X_1)\times(X_k-X_2)\times\ldots\times(X_k-X_{k-1}))\times$$
$$(((X_k\boldsymbol{-}X_{k+i_1})\times\ldots\times(X_k\boldsymbol{-}X_{k+i_p}))\ \boldsymbol{\times}\ ((X_k\boldsymbol{-}X_{k+j_1})\times\ldots\times(X_k\boldsymbol{-}X_{k+j_q})))$$
$$\text{if } r_2>0,$$

$$f(X)=((X_k-X_1)\times(X_k-X_2)\times\ldots\times(X_k-X_{k-1}))\ \boldsymbol{\times}$$
$$(((X_k\boldsymbol{-}X_{k+i_1})\times\ldots\times(X_k\boldsymbol{-}X_{k+i_p}))\ \boldsymbol{\times}\ ((X_k\boldsymbol{-}X_{k+j_1})\times\ldots\times(X_k\boldsymbol{-}X_{k+j_q})))$$
$$\text{if } r_2<0,$$

where $\{i_1,i_2,\ldots,i_p\}$ is a subset of $\{1,2,\ldots,n-k\}$ such that $w(X_{k+i_l})\leq w(X_k)$, $l=1,2,\ldots,p$, and $\{j_1,j_2,\ldots,j_q\}$ is this subset of $\{1,2,\ldots,n-k\}$, for which $w(X_{k+j_l})>w(X_k)$, $l=1,2,\ldots,q$ holds,

and $r_2=|(\overline{x}_k-\overline{x}_{k+1})(\overline{x}_k-\overline{x}_{k+2})\ldots(\overline{x}_k-\overline{x}_n)|-$
$$|(\underline{x}_k-\underline{x}_{k+1})(\underline{x}_k-\underline{x}_{k+2})\ldots(\underline{x}_k-\underline{x}_n)|.$$

iv) $$f(X)=(((X_k-X_{i_1})\times\ldots\times(X_k-X_{i_p}))\ \mathbf{x}\ ((X_k-X_{j_1})\times\ldots\times(X_k-X_{j_q})))$$
$$\times\ ((X_k-X_{k+1})\times\ldots\times(X_k-X_n))\ \text{if}\ r_1>0,$$

$$f(X)=(((X_k-X_{i_1})\times\ldots\times(X_k-X_{i_p}))\ \mathbf{x}\ ((X_k-X_{j_1})\times\ldots\times(X_k-X_{j_q})))$$
$$\mathbf{x}\ ((X_k-X_{k+1})\times\ldots\times(X_k-X_n))\ \text{if}\ r_1<0,$$

where $\{i_1,i_2,\ldots,i_p\}$ is a subset of $\{1,2,\ldots,k-1\}$ such that $w(X_{i_l})\le w(X_k)$, $l=1,2,\ldots,p$, and $\{j_1,j_2,\ldots,j_q\}$ is this subset of $\{1,2,\ldots,k-1\}$, for which $w(X_{j_l})>w(X_k)$, $l=1,2,\ldots,q$ holds, $r_1=|(\bar{x}_k-\bar{x}_1)(\bar{x}_k-\bar{x}_2)\ldots(\bar{x}_k-\bar{x}_{k-1})|-|(\underline{x}_k-\underline{x}_1)(\underline{x}_k-\underline{x}_2)\ldots(\underline{x}_k-\underline{x}_{k-1})|$.

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